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Unboundedness of the Lipschitz Constants of Best Polynomial Approximation

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Let $f \in C[-1, 1]$ be real-valued. We consider the Lipschitz constants $L_n(f)$ of the operators of best uniform polynomial approximation of degree n, $n \in \mathbb{N}$. It is proved that $\limsup_{n \in \mathbb{N}} L_n(f) = \infty$, whenever f is not a polynomial.

1. STATEMENT OF THE RESULT AND NOTATIONS

Let $C[-1, 1]$ denote the set of all real-valued continuous functions on $[-1, 1]$ and let $f \in C[-1, 1]$ be given. We denote by $q_n^*(f)$, $n \in \mathbb{N}$, its best uniform approximation in the set P_n of algebraic polynomials of degree at most $n \in \mathbb{N}$:

$$e_n = e_n(f) := \|f - q_n^*(f)\| := \min_{q \in P_n} \|f - q\| := \min_{q \in P_n} \{ \max_{x \in [-1, 1]} |f(x) - q(x)| \}.$$

By a result of Freud the operator of best polynomial approximation $q_n^*: C[-1, 1] \to P_n$ is pointwise Lipschitz continuous:

Theorem A (Freud [4] or [3], p. 80). *For each $n \in \mathbb{N}$ there exists a constant $L_n(f) < \infty$ such that*

$$\|q_n^*(f) - q_n^*(g)\| \leqslant L_n(f) \, \|f - g\|, \qquad \text{for all} \quad g \in C[-1, 1]. \tag{1}$$

Definition. For each $n \in \mathbb{N}$ we call the smallest constant $L_n(f)$ such that (1) holds the Lipschitz constant of q_n^* at f.

If f is a polynomial it is known that the sequence $(L_n(f))_n$ is bounded ([6], p. 86). As the main result of this paper we shall prove the following conjecture of Henry and Roulier ([6]):

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THEOREM 1. *If f is not a polynomial, then we have*

$$\limsup_{n\to\infty} L_n(f) = \infty.$$

We note that our proof will not provide any concrete estimate for $L_n(f)$.

In what follows we will assume that $f \in C[-1, 1]$ is not a polynomial, and thus $e_n = e_n(f) > 0$ for all $n \in \mathbb{N}_0$. To prove Theorem 1 let

$$E_n = E_n(f) := \{x \in [-1, 1] : |f(x) - q_n^*(f, x)| = e_n\}, \qquad n \in \mathbb{N}_0$$

and

$$\sigma_n(x) = \sigma_n(f, x) := \operatorname{sign}(f(x) - q_n^*(f, x)), \qquad n \in \mathbb{N}_0.$$

We decompose the set E_n into sign components $E_n = \bigcup_{j=1}^m E_n^j$,

$$E_n^1 < E_n^2 < \cdots < E_n^m, \qquad \text{i.e.,} \quad x < y \quad \text{for all } x \in E_n^j, \quad y \in E_n^{j+1},$$

such that $\sigma_n(x)$ is constant on each E_n^j, $1 \leqslant j \leqslant m$, and $m = m(n) = m(f, n)$ is minimal. The Lipschitz constant $L_n(f)$ can be estimated in terms of E_n and $\sigma_n(x)$.

THEOREM 2. *Let $p \in P_n$ be a polynomial satisfying*

$$-1 \leqslant \max_{E_n^j} \sigma_n(x)\, p(x) \leqslant 1, \qquad \textit{for all} \quad 1 \leqslant j \leqslant m(n).$$

Then we have

$$L_n(f) \geqslant \|p\|/2.$$

Proof of Theorem 2. 1. For every given $\alpha > 0$ there exists some $\varepsilon > 0$ such that for $U_\varepsilon(E_n) := \{x \in [-1, 1] : \operatorname{dist}(x, E_n) \leqslant \varepsilon\}$ we have

$$\min_{U_\varepsilon(E_n)} |f(x) - q_n^*(f, x)| \geqslant e_n/2$$

and

$$\max_{U_\varepsilon(E_n)} \sigma_n(x)\, p(x) \leqslant 1 + \alpha.$$

It follows that

$$U_\varepsilon(E_n^j) \cap U_\varepsilon(E_n^k) = \varnothing, \qquad \text{for all} \quad 1 \leqslant j \neq k \leqslant m(n).$$

For some $\delta > 0$ we get

$$|f(x) - q_n^*(f, x)| \leqslant e_n - \delta, \qquad \text{for all} \quad x \in [-1, 1] \backslash U_\varepsilon(E_n).$$

We choose λ such that $0 < \lambda < 1/2 \min\{\delta/\|p\|, e_n/\|p\|, e_n/(1+\alpha)\}$ and obtain

$$|f(x) - q_n^*(f, x) + \lambda p(x)| \leqslant e_n, \qquad \text{for all} \quad x \in [-1, 1] \backslash U_\varepsilon(E_n),$$

and

$$|f(x) - q_n^*(f, x) + \lambda p(x)| \leqslant e_n + (1+\alpha)\lambda, \qquad \text{for all} \quad x \in U_\varepsilon(E_n).$$

Further, for each $1 \leqslant j \leqslant m(n)$, there exists some $x_j \in E_n^j$ such that

$$|f(x_j) - q_n^*(f, x_j) + \lambda p(x_j)| \geqslant e_n - \lambda,$$

and

$$\operatorname{sign}(f(x_j) - q_n^*(f, x_j) + \lambda p(x_j)) = \sigma_n|_{E_n^j}.$$

Thus, for each $1 \leqslant j \leqslant m(n)$, there exists some open non-void interval $I_j \subset U_\varepsilon(E_n^j)$ such that

$$e_n + (1+\alpha)\lambda \geqslant |f(x) - q_n^*(f, x) + \lambda p(x)| \geqslant e_n - (1+\alpha)\lambda, \qquad \text{for all} \quad x \in I_j,$$

and

$$\operatorname{sign}(f(x) - q_n^*(f, x) + \lambda p(x)) = \sigma_n|_{E_n^j}, \qquad \text{for all} \quad x \in I_j.$$

2. To define a suitable function $g = g(n, \alpha) \in C[-1, 1]$ in (1) we put

$$g(x) := f(x), \qquad \text{for all} \quad x \in [-1, 1] \Big\backslash \bigcup_{j=1}^{m(n)} I_j.$$

If $I_j = (a, b)$ and $c := (a+b)/2$, we define g on I_j by:

$$g(a) := f(a), \quad g(b) := f(b), \quad g(c) - q_n^*(f, c) + \lambda p(c) := (e_n + (1+\alpha)\lambda)\, \sigma_n|_{E_n^j},$$

and $g - q_n^*(f) + \lambda p$ is linear in $[a, c]$ and $[c, b]$.

The function $g - q_n^*(f) + \lambda p$ thus has the structure of an error function of best polynomial approximation of degree n, and we obtain

$$q_n^*(g) = q_n^*(f) - \lambda p.$$

By our construction we have

$$\|f - g\| \leqslant 2(1+\alpha)\lambda$$

and

$$\|q_n^*(f)-q_n^*(g)\| = \lambda \, \|p\| \geqslant \frac{\|p\|}{2(1+\alpha)} \, \|f-g\|,$$

which implies $L_n(f) \geqslant \|p\|/2(1+\alpha)$. Since $\alpha > 0$ was arbitrary, Theorem 2 is proved.

Thus, to prove our main result, it will be sufficient to find polynomials $p = p_n \in P_n$, $n \in \mathbb{N}$, satisfying the side condition of Theorem 2 such that $\|p_n\|$ becomes unbounded, as n increases.

We note that for each n, where $q_n^*(f) \neq q_{n+1}^*(f)$, we have $m(n) = n+2$. Since f is not a polynomial, there exists a subsequence L of $\mathbb{N}$ such that $m(n) = n+2$, $n \in L$. In what follows we will only consider the subsequence L. For convenience, we assume that $n \geqslant 4$ for all $n \in L$.

For every $n \in L$ let

$$\xi_j = \xi_j(n) := \min E_n^j \qquad \text{and} \qquad \eta_j = \eta_j(n) := \max E_n^j, \qquad 1 \leqslant j \leqslant n+2,$$

denote the left and right end points of the sign components $E_n^1, ..., E_n^{n+2}$. Since they will always appear in connection with some $n \in L$, we will usually avoid an index n and just write $\xi_1, ..., \xi_{n+2}$ and $\eta_1, ..., \eta_{n+2}$.

For $n \in L$ we consider two types of problems:

Problem $A(n, k, y)$. Let $n \in L$, $k \in \{1, ..., n+2\}$ and $y \in E_n^k$ be fixed. Determine $p_n^k \in P_n$ such that

$$-\sigma_n(y) \, p_n^k(y) \qquad \text{is maximal}$$

subject to the condition that

$$\max_{x \in E_n} \sigma_n(x) \, p_n^k(x) \leqslant 1.$$

By ([2], Lemma 1), each problem $A(n, k, y)$ has a unique solution p_n^k that does not depend on $y \in E_n^k$. Further, there exists a corresponding set of $n+1$ active points $X_n^k = \{x_j \colon 1 \leqslant j \leqslant n+2, \, j \neq k\}$ such that

$$x_j \in E_n^j, \qquad \text{for all} \quad 1 \leqslant j \leqslant n+2, \quad j \neq k,$$

and

$$\sigma_n(x_j) \, p_n^k(x_j) = \max_{E_n^j} \sigma_n(x) \, p_n^k(x) = 1, \qquad \text{for all} \quad 1 \leqslant j \leqslant n+2, \quad j \neq k. \tag{2}$$

A solution of $A(n, k, y)$ will in general not satisfy the side condition of Theorem 2. Therefore, we will mainly consider the

Problem $B(n, k)$. Let $n \in L$, $k \in \{1, ..., n+1\}$ and $y := (\eta_k + \xi_{k+1})/2$ be fixed.

Determine $p_n^k \in P_n$ such that

$$-\sigma_n|_{E_n^k}\, p_n^k(y) \qquad \text{is maximal}$$

subject to the condition that

$$-1 \leqslant \max_{E_n^j} \sigma_n(x)\, p_n^k(x) \leqslant 1, \qquad \text{for all} \quad 1 \leqslant j \leqslant n+2.$$

By a simple compactness argument, each problem $B(n, k)$ has a solution $p_n^k \in P_n$. We will leave aside the question of uniqueness and merely choose a solution p_n^k for each $n \in L$ and $1 \leqslant k \leqslant n+1$.

Looking at a suitable interpolating polynomial $p \in P_2$:

$$p(\xi_k) = p(\xi_{k+1}) = -\sigma_n|_{E_n^k}, \qquad \text{and} \qquad p(1) = \sigma_n|_{E_n^k} \ \text{ or } \ p(-1) = \sigma_n|_{E_n^k},$$

we easily get that the optimal value of $B(n, k)$ satisfies

$$-\sigma_n|_{E_n^k}\, p_n^k(y) > 1, \qquad \text{for all} \quad 1 \leqslant k \leqslant n+1, \quad n \in L.$$

Moreover, for the solution p_n^k of $B(n, k)$ there exists a corresponding set X_n^k of $n+1$ active points:

THEOREM 3. *Let p_n^k be a solution of $B(n, k)$, $n \in L$, $1 \leqslant k \leqslant n+1$. Then there exists some index $j_k = j_k(n) \in \{1, ..., n+2\}$ such that for $I_k = I_k(n) := \{1, ..., n+2\} \setminus \{j_k\}$ the following assertions hold*:

(a) *For each $j \in I_k$ there exists some $x_j \in E_n^j$ such that*

$$\sigma_n(x_j)\, p_n^k(x_j) = \max_{E_n^j} \sigma_n(x)\, p_n^k(x) \in \{-1, 1\}.$$

(b) *We have $\{k, k+1\} \subset I_k$ and $x_k = \xi_k$, $x_{k+1} = \xi_{k+1}$.*

(c) *If we renumber the set $X_n^k := \{x_j,\ j \in I_k\} = \{x'_1 < \cdots < x'_{n+1}\}$ by consecutive indices, then*

$$p_n^k(x'_j) = -p_n^k(x'_{j+1}), \qquad \text{whenever} \quad (x'_j, x'_{j+1}) \neq (x_k, x_{k+1}) = (\xi_k, \xi_{k+1}),$$

and for $y = (\eta_k + \xi_{k+1})/2$

$$p_n^k(\xi_k) = p_n^k(\xi_{k+1}) = \operatorname{sign}(p_n^k(y)) = -\sigma_n|_{E_n^k}.$$

Proof of Theorem 3. Let p_n^k be a solution of $B(n,k)$. We define

$$I := \{1 \leqslant j \leqslant n+2 : \max_{E_n^j} \sigma_n(x)\, p_n^k(x) \in \{-1, 1\}\},$$

and choose $x_j \in E_n^j$, $j \in I$, such that

$$\sigma_n(x_j)\, p_n^k(x_j) = \max_{E_n^j} \sigma_n(x)\, p_n^k(x).$$

We introduce an additional index $k_y := k + 1/2$ corresponding to the point $y = (\eta_k + \xi_{k+1})/2$ and define a sign function σ on the set $I \cup \{k_y\}$ by

$$\sigma(j) := \operatorname{sign}(p_n^k(x_j)), \qquad j \in I,$$
$$\sigma(k_y) := -\operatorname{sign}(p_n^k(y)).$$

Next, we decompose $I \cup \{k_y\}$ into sign components $I \cup \{k_y\} = \bigcup_{j=1}^m I^j$

$$I^1 < \cdots < I^m,$$

such that σ is constant on each I^j and m is minimal.

If we assume that $m \leqslant n+1$, then there exists a polynomial $p \in P_n$ such that

$$\operatorname{sign}(p(x)) = \operatorname{sign}(p_n^k(x_j)) \qquad \text{for all} \quad x \in E_n^j, \quad j \in I,$$

and

$$\operatorname{sign}(p(y)) = -\operatorname{sign}(p_n^k(y)).$$

For some suitable $\lambda > 0$ the polynomial $p_n^k - \lambda p \in P_n$ satisfies the side conditions of $B(n,k)$ and has $|p_n^k(y) - \lambda p(y)| > |p_n^k(y)|$, which yields a contradiction.

It follows that $m \geqslant n+2$. Hence, we may choose a subset $\{y_1 < \cdots < y_{n+1}\} \subset \{x_j, j \in I\}$ such that

$$p_n^k(y_j) = -p_n^k(y_{j+1}), \qquad \text{whenever} \quad y \notin (y_j, y_{j+1}),$$
$$p_n^k(y_j) = p_n^k(y_{j+1}) = \operatorname{sign}(p_n^k(y)) = -\sigma_n|_{E_n^k}, \qquad \text{if} \quad y \in (y_j, y_{j+1}).$$

If we assume that $y < y_1$ then we must have $k = 1$ and $y_1 \in E_n^2$. Further, there must be n zeros of p_n^1 in (y_1, y_{n+1}), and therefore all local extrema of p_n^1 are in (y_1, y_{n+1}). Since $\min_{E_n^1} |p_n^1(x)| \leqslant 1$, $|p_n^1(y)| > 1$ and $|p_n^1(y_1)| = 1$, there must be a local extremum in (ξ_1, y_1) and we obtain a contradiction. It follows in the same way that $y > y_n + 1$ can not hold.

Therefore, there exists a subinterval (y_j, y_{j+1}) containing the point y. In each of the $n-1$ subintervals where $y \notin (y_j, y_{j+1})$ there must be a zero of p_n^k and it is not difficult to see that there are no further zeros in (y_1, y_{n+1}). In case p_n^k has exact degree n, there is one additional zero in $\mathbb{R} \backslash (y_1, y_{n+1})$.

Between each pair of consecutive zeros there is exactly one local extremum of p_n^k and besides these there are no further local extrema.

Now, let (y_j, y_{j+1}) be the subinterval such that $y \in (y_j, y_{j+1})$. It follows that p_n^k has no zero in $[y_j, y_{j+1}]$ and that $p_n^k(y_j) = \operatorname{sign}(p_n^k(y)) = -\sigma_n|_{E_n^k} = p_n^k(y_{j+1})$. Hence, there is exactly one local extremum in (y_j, y_{j+1}). Since $\min_{E_n^k} |p_n^k|, \min_{E_n^{k+1}} |p_n^k| \leqslant 1$, this implies that $y_j \in E_n^k$ and $y_{j+1} \in E_n^{k+1}$, and thus

$$\sigma_n(y_j)\, p_n^k(y_j) = \max_{E_n^k} \sigma_n(x)\, p_n^k(x) = -1$$

and

$$\sigma_n(y_{j+1})\, p_n^k(y_{j+1}) = \max_{E_n^{k+1}} \sigma_n(x)\, p_n^k(x) = 1.$$

From here, it is easy to see that $y_{j+1} = \xi_{k+1}$.

If we assume that $\xi_k < y_j \in E_n^k$ then, since $\sigma_n(\xi_k)\, p_n^k(\xi_k) \leqslant -1$ and $\sigma_n(y_j)\, p_n^k(y_j) = -1$, there must be at least two local extrema in $[\xi_k, y_{j+1}]$ and therefore at least one zero in $[\xi_k, y_{j+1}]$, and thus in $[\xi_k, y_j]$. This implies that there are at least two zeros in $[\xi_k, y_j]$, which yields a contradiction to the position of the zeros of p_n^k derived above.

Choosing $x'_1 := y_1, ..., x'_{n+1} := y_{n+1}$ and defining $I_k = \{1, ..., n+2\} \backslash \{j_k\}$ by $\{x_j, j \in I_k\} := \{x'_1, ..., x'_{n+1}\}$, Theorem 3 is proved.

We introduce some notations and collect some simple properties for the solutions p_n^k and $X_n^k = \{x_j, j \in I_k\}$ of a given problem $B(n, k)$, $n \in L$, $1 \leqslant k \leqslant n+1$.

(a) In what follows we will use both notations $X_n^k = \{x_j, j \in I_k\}$ and $X_n^k = \{x'_1 < \cdots < x'_{n+1}\}$ for the solution of $B(n, k)$.

For the sake of simplicity we shall usually avoid an index n for $I_k = I_k(n)$ and $j_k = j_k(n)$ as well as n or k for the points $x_j = x_j(n, k)$, $x'_j = x'_j(n, k) \in X_n^k$.

(b) Let $x'_\nu = x_j \in E_n^j \subset [\xi_j, \eta_j]$, where $1 \leqslant \nu \leqslant n+1$. Then we put $\xi'_\nu := \xi_j$ and $\eta'_\nu := \eta_j$, i.e., $x'_\nu \in [\xi'_\nu, \eta'_\nu]$. We note that $j = \nu$ if $j < j_k$, and $j = \nu + 1$ if $j > j_k$.

Again, we will usually avoid indices n or k for the points $\xi'_\nu = \xi'_\nu(n, k)$ and $\eta'_\nu = \eta'_\nu(n, k)$, $1 \leqslant \nu \leqslant n+1$.

Further, let $k' = k'(n, k)$ be defined by $x'_{k'} = x_k = \xi_k$. We get $k' = k-1$ or $k' = k$.

(c) With these notations, the following properties of a solution p_n^k and the corresponding active sets I_k, $X_n^k = \{x_j,\ j \in I_k\} = \{x'_1 < \cdots < x'_{n+1}\}$ may easily be derived from Theorem 3:

The polynomial p_n^k has exactly $n-1$ zeros $\zeta_1, ..., \zeta_{k'-1}, \zeta_{k'+1}, ..., \zeta_n$ in (x'_1, x'_{n+1}), where exactly one zero is in every subinterval (x'_j, x'_{j+1}), $1 \leqslant j \leqslant n$, except for the subinterval $(x'_{k'}, x'_{k'+1}) = (x_k, x_{k+1}) = (\xi_k, \xi_{k+1})$:

$$x'_1 < \zeta_1 < x'_2 < \cdots < \zeta_{k'-1} < x'_{k'} := x_k = \xi_k < \xi_{k+1}$$

$$\xi_{k+1} = x_{k+1} =: x'_{k'+1} < \zeta_{k'+1} < \cdots < x'_n < \zeta_n < x'_{n+1}.$$

In case p_n^k has exact degree n, there exists one additional zero $\zeta_0 \in \mathbb{R} \setminus [x'_1, x'_{n+1}]$. We shall avoid an index n or k for the zeros $\zeta_j = \zeta_j(n, k)$.

(d) Further, we may divide I_k into at most three blocks of consecutive indices such that the value of $\max_{E_n^j} \sigma_n(x)\, p_n^k(x)$ is constant for j in each of the blocks. If, for example, $j_k < k$ then we have

$$\max_{E_n^j} \sigma_n(x)\, p_n^k(x) = 1, \qquad \text{for all} \quad 1 \leqslant j < j_k,$$

$$\max_{E_n^j} \sigma_n(x)\, p_n^k(x) = -1, \qquad \text{for all} \quad j_k < j \leqslant k,$$

$$\max_{E_n^j} \sigma_n(x)\, p_n^k(x) = 1, \qquad \text{for all} \quad k+1 \leqslant j \leqslant n+2.$$

2. PROOF OF THEOREM 1

We assume that $(L_n(f))_n$ is bounded. By Theorem 2 this implies that for some constant $M < \infty$, not depending on $n \in L$, we have

$$\|p\| \leqslant M \tag{3}$$

and thus, by the Bernstein inequality ([8], p. 118),

$$|p'(x)| \leqslant \frac{n}{\sqrt{1-x^2}} M, \qquad x \in [-1, 1], \tag{4}$$

for any polynomial $p \in P_n$, $n \in L$, satisfying the side condition

$$-1 \leqslant \max_{E_n^j} \sigma_n(x)\, p(x) \leqslant 1, \qquad \text{for all} \quad 1 \leqslant j \leqslant n+2.$$

In particular, (3) and (4) hold for the solution p_n^k of any of the problems $B(n, k)$, $n \in L$, $1 \leqslant k \leqslant n+1$.

Our proof turns out to be elementary but quite technical. Therefore it is split into several lemmas, which are valid under the assumption that (3) holds and which will finally lead to a contradiction.

Throughout the proof, C and D are used to denote absolute positive constants that may depend only on the constant M in (3). We note that the symbols C, D used in different lines of the proof may have different values.

In a first step we control the distances between the various points induced by the sets E_n^j and the problems $B(n, k)$. To this aim we introduce "standard distances" $d_n(j, v)$ that behave approximately like $|\cos(j\pi/n) - \cos(v\pi/n)|$ for $j \neq v$.

Lemma 1. *For each $n \in \mathbb{N}$ let*

$$d_n(j) := \frac{\min\{j, n+3-j\}}{n^2}, \qquad 1 \leqslant j \leqslant n+2$$

and $$d_n(j, v) = d_n(v, j) = \sum_{l=j}^{v} d_n(l), \qquad 1 \leqslant j \leqslant v \leqslant n+2.$$

Then there exist constants $C_1, D_1 > 0$, not depending on $n \in L$ or $1 \leqslant k \leqslant n+1$, such that the following statements hold.

Let p_n^k and $X_n^k = \{x_j, j \in I_k\} = \{x'_1 < \cdots < x'_{n+1}\}$ be a solution of $B(n, k)$ and $\zeta_1, ..., \zeta_{k'-1}, \zeta_{k'+1}, ..., \zeta_n$ denote the zeros of p_n^k in $[x'_1, x'_{n+1}]$.

Further, let

$$Y_j = Y_j(n, k) := \{x_j, \xi_j, \eta_j, x'_j, \xi'_j, \eta'_j, \zeta_j\}, \qquad \textit{for all} \quad 1 \leqslant j \leqslant n+2,$$

where each of the points $x_j, x'_j, \xi'_j, \eta'_j, \zeta_j$ occurs only if it is defined in connection with p_n^k, X_n^k.

(a) *For all $1 \leqslant j, v \leqslant n+2$ and all $y_j \in Y_j$ and $y_v \in Y_v$ we have*

$$|y_j - y_v| \leqslant C_1 d_n(j, v).$$

(b) *For all $1 \leqslant j, v \leqslant n+2$ and all $y_j \in Y_j$ and $y_v \in Y_v$ we have*

$$|y_j - y_v| \geqslant D_1 d_n(j, v),$$

whenever there exists a point $x \in X_n^k$ and a zero $\zeta \in [-1, 1]$ of p_n^k both between y_j and y_v.

Proof of Lemma 1. Let μ denote the arcsine distribution of $[-1, 1]$, i.e.,

$$d\mu(x) = \frac{2}{\pi} \frac{1}{\sqrt{1-x^2}}, \qquad x \in [-1, 1].$$

1. For every $n \in L$ we consider the partitions $\xi_0 := -1, \xi_1, ..., \xi_{n+2}, \xi_{n+3} := 1$ induced by the left end-points of $E_n^1, ..., E_n^{n+2}$. We show that there exists some $C > 0$, not depending on $n \in L$, such that

$$\mu([\xi_j, \xi_{j+1}]) \leqslant \frac{C}{n}, \qquad \text{for all} \quad 0 \leqslant j \leqslant n+2.$$

If not, we may find closed intervals, say $J_n \subset [-1, 1]$, $n \in L$, such that $E_n^\nu \backslash J_n \neq \emptyset$ for all $1 \leqslant \nu \leqslant n+2$, and $\limsup_{n \in L} n\mu(J_n) = \infty$.

Then, by a slight modification of the proof of ([1], Theorem 6), there exist polynomials $q_n \in P_n$ such that $q_n(x) \neq 0$ for $x \in J_n$ and

$$\liminf_{n \in L} \Big(\sup_{x \in [-1, 1] \backslash J_n} |q_n(x)| \Big) / \|q_n\| = 0.$$

It is easy to see that this contradicts our principal assumption (3).

2. Let p_n^k and X_n^k be a solution of $B(n, k)$, $n \in L$, $1 \leqslant k \leqslant n+1$. It follows by (4) that there exists some $D > 0$, not depending on $n \in L$ or $1 \leqslant k \leqslant n+1$, such that for all $x \in X_n^k$ and each zero $\zeta \in [-1, 1]$ of p_n^k we have

$$\mu([\min(\zeta, x), \max(\zeta, x)]) \geqslant \frac{D}{n}.$$

3. For every $n \in L$ and $1 \leqslant k \leqslant n+1$ we recall the interlacing properties of the end points of $E_n^1, ..., E_n^{n+2}$ and the points x_j, ζ_j induced by the solution of $B(n, k)$.

Let $y_j \in Y_j$, $y_\nu \in Y_\nu$, where $y_j = \cos(\phi_j) \leqslant y_\nu = \cos(\phi_\nu)$ and $\phi_j, \phi_\nu \in [0, \pi]$. By the first part, it is easy to see that for some $C > 0$, not depending on $1 \leqslant j, \nu \leqslant n+2$ or $n \in L$, we have

$$\mu([y_j, y_\nu]) \leqslant C \frac{|\nu - j| + 1}{n},$$

and thus for some $C > 0$

$$|\phi_\nu - \phi_j| \leqslant C \frac{|\nu - j| + 1}{n}.$$

In the same way it follows that

$$|\pi-\phi_j|\leqslant C\frac{j}{n}, \qquad |\phi_j-0|\leqslant C\frac{n+3-j}{n},$$

$$|\pi-\phi_\nu|\leqslant C\frac{\nu}{n}, \qquad |\phi_\nu-0|\leqslant C\frac{n+3-\nu}{n}.$$

Suppose that there exist $x\in X_n^k$ and a zero ζ of p_n^k such that $y_j\leqslant x, \zeta\leqslant y_\nu$. Then, by the second part, there exists some $D>0$, not depending on j, ν or $n\in L$ such that

$$\mu([y_j, y_\nu])\geqslant D\frac{|\nu-j|+1}{n},$$

$$\mu([-1, y_\nu])\geqslant D\frac{\nu}{n}, \qquad \mu([y_j, 1])\geqslant D\frac{n+3-j}{n}.$$

Therefore, we have for some $D>0$

$$|\phi_\nu-\phi_j|\geqslant D\frac{|\nu-j|+1}{n},$$

$$|\pi-\phi_\nu|\geqslant D\frac{\nu}{n}, \qquad |\phi_j-0|\geqslant D\frac{n+3-j}{n}.$$

The estimates stated in Lemma 1 now follow by elementary but somewhat lengthy calculations from

$$|y_\nu-y_j|=\left|\int_{\phi_j}^{\phi_\nu}\sin(t)\,dt\right|.$$

The following Lemma 2 provides some general estimates on sums of the $d_n(j,\nu)$ introduced in Lemma 1.

LEMMA 2. *There exist $C_2, D_2>0$ not depending on $n\geqslant 2$ or $1\leqslant j, l\leqslant n+2$ such that*

$$\text{(a)}\quad D_2\log(n)\leqslant\sum_{\substack{\nu=1\\ \nu\neq j}}^{n+2}\frac{d_n(\nu)}{d_n(j,\nu)}\leqslant C_2\log(n),$$

$$\text{(b)}\quad \sum_{\substack{\nu=1\\ \nu\neq j}}^{n+2}\frac{d_n(j)}{d_n(j,\nu)}\leqslant C_2\log(n),$$

$$\text{(c)} \quad \sum_{v=2}^{n+1} \frac{d_n(v)}{\sqrt{d_n(1, v)\, d_n(v, n+2)}} \leqslant C_2,$$

$$\text{(d)} \quad \sum_{\substack{v=1 \\ v \neq j}}^{n+2} \frac{d_n(v)^2}{d_n(j, v)^2}, \sum_{\substack{v=1 \\ v \neq j}}^{n+2} \frac{d_n(v)\, d_n(j)}{d_n(j, v)^2} \leqslant C_2,$$

$$\text{(e)} \quad \sum_{l < v < j} \frac{d_n(v)}{d_n(j, v)} \geqslant D_2 \log(j - l) - C_2.$$

Proof of Lemma 2. The proofs of parts (a), (b), (c), (d) are given in ([5], Lemma 2). Part (e) follows similarly to part (a).

We show that for the solutions $X_n^k = \{x_j,\, j \in I_k\}$ of $B(n, k)$ the products $\prod_{j \in I_k, j \neq k} |x_k - x_j|$ become uniformly small, as $n \in L$ increases.

LEMMA 3. *There exist constants $\delta > 0$ and $C_3 > 0$, not depending on $n \in L$ or $3 \leqslant k \leqslant n$, such that for the solution $X_n^k = \{x_j,\, j \in I_k\} = \{x'_1 < \cdots < x'_{n+1}\}$ of $B(n, k)$ we have*

$$\prod_{\substack{j \in I_k \\ j \neq k}} |x_k - x_j| = \prod_{\substack{j=1 \\ j \neq k'}}^{n+1} |x_k - x'_j| \leqslant \frac{C_3}{n^\delta} \frac{|x_k - x_{k+1}|}{d_n(k)^2} \frac{1}{2^n}.$$

Proof of Lemma 3. The solution p_n^k of $B(n, k)$ has exactly $n-1$ zeros $\zeta_1, ..., \zeta_{k'-1}, \zeta_{k'+1}, ..., \zeta_n$ in $[x'_1, x'_{n+1}]$ which are ordered in the following way

$$x'_1 < \zeta_1 < x'_2 < \cdots < \zeta_{k'-1} < x'_{k'} = x_k = \xi_k < \xi_{k+1}$$

$$\xi_{k+1} = x_{k+1} = x'_{k'+1} < \zeta_{k'+1} < \cdots < x'_n < \zeta_n < x'_{n+1}.$$

In case p_n^k has exact degree n, there is one additional zero $\zeta_0 \notin [x'_1, x'_{n+1}]$.

1. We show that for some $C > 0$, not depending on $n \in L$ or $3 \leqslant k \leqslant n$, we have

$$|(x_k - x'_1)(x_k - x'_{n+1})| \prod_{\substack{j=1 \\ j \neq k'}}^{n} |x_k - \zeta_j| \leqslant C \frac{1}{2^n}.$$

For the sake of shortness we give the computation only for the case that p_n^k has exact degree n. In case p_n^k has exact degree $n-1$, we just have to leave out the terms appearing in $\{\cdots\}$. If $p_n^k(x) = a_n^k x^n + \cdots$, $a_n^k \neq 0$, then the monic polynomial

$$q(x) := \frac{p_n^k(x)}{a_n^k \{x - \zeta_0\}} \in P_{n-1}$$

has alternating signs at the n points x_l of $X_n^k \backslash \{x_k\}$ and absolute value

$$|q(x_l)| = \frac{1}{|a_n^k \{x_l - \zeta_0\}|} \geqslant \frac{1}{|a_n^k| \{|\zeta_0| + 1\}}.$$

It follows from extremal properties of the Chebyshev polynomials ([3], p. 77) that

$$\frac{1}{|a_n^k| \{|\zeta_0| + 1\}} \leqslant \frac{1}{2^{n-2}}$$

and, therefore,

$$\begin{aligned}
&|(x_k - x_1')(x_k - x_{n+1}')| \prod_{\substack{j=1 \\ j \neq k'}}^{n} |x_k - \zeta_j| \\
&= |(x_k - x_1')(x_k - x_{n+1}')| \frac{|p_n^k(x_k)|}{|a_n^k| \{|x_k - \zeta_0|\}} \\
&= |(x_k - x_1')(x_k - x_{n+1}')| \frac{1}{|a_n^k| \{|x_k - \zeta_0|\}} \\
&\leqslant |(x_k - x_1')(x_k - x_{n+1}')| \left\{ \frac{|\zeta_0| + 1}{|x_k - \zeta_0|} \right\} \frac{1}{2^{n-2}} \\
&\leqslant C \frac{1}{2^n},
\end{aligned}$$

for some $C > 0$ which, in particular, does not depend on the position of $\zeta_0 \notin [x_1', x_{n+1}']$.

2. We estimate

$$\left\{\prod_{\substack{j=2\\ j\neq k'}}^{n} |x_k - x'_j|\right\}\left\{\prod_{\substack{j=1\\ j\neq k'}}^{n} |x_k - \zeta_j|\right\}^{-1}$$

$$=\frac{|x_k - x_{k+1}|}{|(x_k - \zeta_{k'-1})(x_k - \zeta_{k'+1})|}\prod_{j=2}^{k'-1}\frac{|x_k - x'_j|}{|x_k - \zeta_{j-1}|}\prod_{j=k'+2}^{n}\frac{|x_k - x'_j|}{|x_k - \zeta_j|}$$

$$=\frac{|x_k - x_{k+1}|}{|(x_k - \zeta_{k'-1})(x_k - \zeta_{k'+1})|}\prod_{j=2}^{k'-1}\left(1-\frac{|x'_j - \zeta_{j-1}|}{|x_k - \zeta_{j-1}|}\right)$$

$$\times\prod_{j=k'+2}^{n}\left(1-\frac{|x'_j - \zeta_j|}{|x_k - \zeta_j|}\right).$$

By Lemma 1 we have

$$\frac{|x_k - x_{k+1}|}{|(x_k - \zeta_{k'-1})(x_k - \zeta_{k'+1})|}\leqslant\frac{|x_k - x_{k+1}|}{D_1^2\, d_n(k, k'-1)\, d_n(k, k'+1)}\leqslant\frac{|x_k - x_{k+1}|}{D_1^2 d_n(k)^2}.$$

Further, Lemma 1 and Lemma 2(a) yield that

$$\prod_{j=2}^{k'-1}\left(1-\frac{|x'_j - \zeta_{j-1}|}{|x_k - \zeta_{j-1}|}\right)\prod_{j=k'+2}^{n}\left(1-\frac{|x'_j - \zeta_j|}{|x_k - \zeta_j|}\right)$$

$$\leqslant \exp\left\{-\sum_{j=2}^{k'-1}\frac{|x'_j - \zeta_{j-1}|}{|x_k - \zeta_{j-1}|}-\sum_{j=k'+2}^{n}\frac{|x'_j - \zeta_j|}{|x_k - \zeta_j|}\right\}$$

$$\leqslant \exp\left\{-\frac{D_1}{C_1}\left(\sum_{j=2}^{k'-1}\frac{d_n(j, j-1)}{d_n(k, j-1)}+\sum_{j=k'+2}^{n}\frac{d_n(j)}{d_n(k, j)}\right)\right\}$$

$$\leqslant\frac{C}{n^\delta},$$

for some suitable $C, \delta > 0$, not depending on $n \in L$ or $3 \leqslant k \leqslant n$.

3. Putting part one and part two together, we obtain that

$$\prod_{\substack{j=1\\ j\neq k}}^{n+1} |x_k - x_j| \leqslant \frac{C}{n^\delta}\frac{|x_k - x_{k+1}|}{d_n(k)^2}\frac{1}{2^n},$$

for some $C =: C_3 > 0$, and Lemma 3 is proved.

Next, we consider a sequence of sign components E_n^k, where $k = k(n)$, $n \in L$. We provide a sufficient condition to control the distances of E_n^k to E_n^{k-1} and E_n^{k+1} from below, as $n \in L$ increases.

Lemma 4. *There exists some $D_4>0$ with the following property:*
Let $k=k(n)\in\{3, ..., n\}$, $n\in L$, be a sequence of indices such that for every $\varepsilon>0$

$$\liminf_{n\in L} n^{\varepsilon}2^n\, d_n(k)^3 \prod_{\substack{j=1\\|j-k|\geqslant 2}}^{n+2} |\xi_k-\xi_j|>0.$$

Then we have

$$|\xi_k-\eta_{k-1}|, |\xi_{k+1}-\eta_k|>D_4 d_n(k), \qquad \textit{for all} \quad k=k(n), \quad n\in L.$$

Proof of Lemma 4. For $n\in L$ and $k=k(n)$ let us now denote by p_n^k the solution of $A(n, k, \xi_k)$, and by $X_n^k=\{x_j\in E_n^j : 1\leqslant j\leqslant n+2, j\neq k\}$ the corresponding set of active points:

$$\sigma_n(x_j)\, p_n^k(x_j)=\max_{E_n^j} \sigma_n(x)\, p_n^k(x)=1, \qquad \text{for all} \quad 1\leqslant j\leqslant n+2, \quad j\neq k.$$

We avoid an index n or $k=k(n)$ for the $x_j=x_j(n, k)\in X_n^k$.

From the properties (2) of the solution of a problem $A(n, k, y)$ it is not difficult to see that

$$p_n^k(\eta_{k-1})=p_n^k(\xi_{k+1})=-\sigma_n|_{E_n^k},$$

and

$$|p_n^k(x)|\geqslant 1, \qquad \text{for all} \quad x\in[\eta_{k-1}, \xi_{k+1}].$$

1. We first show that $\lim_{n\in L}\|p_n^{k(n)}\|=\infty$.

Assume that $\|p_n^k\|$, $k=k(n)$, $n\in L'$, remains bounded for a subsequence L' of L. Then, ([5], Lemma 3) yields that for some δ, $C>0$ not depending on $n\in L'$ we must have

$$\prod_{\substack{j=1\\|j-k|\geqslant 2}}^{n+2} |\xi_k-x_j|\leqslant\frac{C}{n^{\delta}}\frac{1}{d_n(k)^2}\frac{1}{2^n}, \qquad \text{for all} \quad n\in L'.$$

We note that the boundedness of $\|p_n^k\|$, $k=k(n)$, $n\in L'$, together with Lemma 1 is sufficient to prove this relation, although ([5], Lemma 3) is stated under the assumption that the solutions of all possible problems $A(n, j, y)$ remain uniformly bounded.

On the other hand, since $|\xi_k - x_j| \geqslant |\xi_k - \xi_{j+1}|$ for $j \leqslant k-3$, we have

$$\prod_{\substack{j=1\\|j-k|\geqslant 2}}^{n+2} |\xi_k - x_j| \geqslant \prod_{j=1}^{k-2} |\xi_k - x_j| \prod_{j=k+2}^{n+2} |\xi_k - \xi_j|$$

$$\geqslant |\xi_k - x_{k-2}| \prod_{j=2}^{k-2} |\xi_k - \xi_j| \prod_{j=k+2}^{n+2} |\xi_k - \xi_j|$$

$$\geqslant \frac{|\xi_k - x_{k-2}|}{|\xi_k - \xi_1|} \prod_{\substack{j=1\\|j-k|\geqslant 2}}^{n+2} |\xi_k - \xi_j|$$

$$\geqslant \frac{|\xi_k - x_{k-2}|}{2} \prod_{\substack{j=1\\|j-k|\geqslant 2}}^{n+2} |\xi_k - \xi_j|.$$

We assumed that $\|p_n^k\|$, $k = k(n)$, $n \in L'$, is bounded, and we have $|p_n^k(x_{k-2}) - p_n^k(x_{k-1})| = 2$. The Bernstein inequality thus yields that for some $D > 0$

$$\mu([x_{k-2}, x_{k-1}]) \geqslant \frac{D}{n}, \qquad \text{for all} \quad k = k(n), \quad n \in L',$$

where μ denotes the arcsine distribution of $[-1, 1]$. By the arguments of Lemma 1 we therefore get that for some $D > 0$, not depending on $n \in L'$, we have

$$|\xi_k - x_{k-2}| \geqslant |x_{k-1} - x_{k-2}| \geqslant D d_n(k),$$

and thus

$$\prod_{\substack{j=1\\|j-k|\geqslant 2}}^{n+2} |\xi_k - x_j| \geqslant \frac{D d_n(k)}{2} \prod_{\substack{j=1\\|j-k|\geqslant 2}}^{n+2} |\xi_k - \xi_j|.$$

Choosing $\varepsilon = \delta/2$ in the assumption of Lemma 4, we obtain a contradiction.

2. If we put

$$m_n := \min_{E_n^k} |p_n^k(x)|, \qquad \text{for all} \quad k = k(n), \quad n \in L,$$

then $m_n \geqslant 1$ and the polynomial p_n^k / m_n satisfies the side condition of Theorem 2:

$$-1 \leqslant \max_{E_n^j} \frac{p_n^k(x)}{m_n} \sigma_n(x) \leqslant 1, \qquad \text{for all} \quad 1 \leqslant j \leqslant n+2.$$

It follows by (3) that

$$\frac{\|p_n^k\|}{m_n} \leqslant M, \qquad \text{for all} \quad n \in L,$$

and the first part implies that $\lim_{n \in L} m_n = \infty$.

By the Bernstein-inequality (4) we have

$$\left|\frac{(p_n^k)'(x)}{m_n}\right| \leqslant \frac{n}{\sqrt{1-x^2}} M, \qquad \text{for all} \quad x \in [-1, 1].$$

Since $|p_n^k(\xi_k)|/m_n$, $|p_n^k(\eta_k)|/m_n \geqslant 1$ and $|p_n^k(\eta_{k-1})|/m_n = |p_n^k(\xi_{k+1})|/m_n = 1/m_n$ tend to zero, there exists some $D > 0$ such that

$$\mu([\eta_{k-1}, \xi_k]), \mu([\eta_k, \xi_{k+1}]) \geqslant \frac{D}{n}, \qquad \text{for all} \quad k = k(n), \quad n \in L.$$

By the arguments of Lemma 1, we finally get that for some $D_4 > 0$

$$|\xi_k - \eta_{k-1}|, |\xi_{k+1} - \eta_k| > D_4\, d_n(k), \qquad \text{for all} \quad k = k(n), \quad n \in L.$$

Next, we will prepare the construction of a special sequence of indices fulfilling the assumptions of Lemma 4. It will be finally given in Lemma 8.

In Lemmas 5, 6, and 7 we consider the solutions $X_n^1 = \{x_j, j \in I_1\} = \{x'_1 < \cdots < x'_{n+1}\}$ of the special problems $B(n, 1)$, $n \in L$. We note that, by Theorem 3, $x_1 = x'_1 = \xi_1$ and $x_2 = x'_2 = \xi_2$ for all $n \in L$.

First, we show that most of the products $\prod_{\nu=1, \nu \neq j}^{n+1} |x'_j - x'_\nu|$, $1 \leqslant j \leqslant n+1$, do not become too small for our solution X_n^1, as $n \in L$ increases.

LEMMA 5. *Suppose that* $\varepsilon > 0$. *For the solutions* $X_n^1 = \{x_j, j \in I_1\} = \{x'_1 < \cdots < x'_{n+1}\}$ *of* $B(n, 1)$, $n \in L$, *let* $a_n(\varepsilon)$ *denote the number of indices* $3 \leqslant j \leqslant n+1$ *such that*

$$\prod_{\substack{\nu=1 \\ \nu \neq j}}^{n+1} |x'_j - x'_\nu| \leqslant \frac{1}{n^\varepsilon} \frac{1}{d_n(j)} \frac{1}{2^n}.$$

Then we have

$$\lim_{n \in L} \frac{a_n(\varepsilon)}{n} = 0.$$

Proof of Lemma 5. Suppose that there exists some $a>0$ and a subsequence L' of L such that

$$\frac{a_n(\varepsilon)}{n} \geqslant a \qquad \text{for all} \quad n \in L'.$$

For some $C=C(a)>0$ we have that for all sufficiently large $n \in L$

$$1/d_n(j) \leqslant Cn, \qquad \text{for all} \quad an/4 \leqslant j \leqslant (n+2)-an/4.$$

It follows that among the indices $an/4 \leqslant j \leqslant (n+1)-an/4$ there exist at least $an/4$ indices j such that

$$\prod_{\substack{v=1\\ v\neq j}}^{n+1} |x'_j - x'_v| \leqslant \frac{C}{n^\varepsilon} n \frac{1}{2^n} \qquad \text{for all sufficiently large} \quad n \in L'.$$

If a_n^1 denotes the leading coefficient of p_n^1, then the sign structure of $p_n^1(x'_j)$, $1 \leqslant j \leqslant n+1$:

$$p_n^1(x'_1) = p_n^1(x'_2) \qquad \text{and} \qquad p_n^1(x'_j) = -p_n^1(x'_{j+1}), \qquad 2 \leqslant j \leqslant n,$$

and the Lagrange interpolation formula yield that

$$|a_n^1| = \left| \frac{-1}{\prod_{v=1,\, v\neq 1}^{n+1} |x'_1 - x'_v|} + \sum_{j=2}^{n+1} \frac{1}{\prod_{v=1,\, v\neq j}^{n+1} |x'_j - x'_v|} \right|.$$

Since

$$\prod_{\substack{v=1\\ v\neq 1}}^{n+1} |x'_1 - x'_v| > \prod_{\substack{v=1\\ v\neq 2}}^{n+1} |x'_2 - x'_v|,$$

we get for all sufficiently large $n \in L'$

$$|a_n^1| \geqslant \sum_{j=3}^{n+1} \frac{1}{\prod_{v=1,\, v\neq j}^{n+1} |x'_j - x'_v|} \geqslant \frac{an}{4} \frac{1}{C} \frac{n^\varepsilon}{n} 2^n \geqslant \frac{a}{4C} n^\varepsilon 2^n.$$

It follows that

$$\|p_n^1\| \geqslant |a_n^1| \frac{1}{2^{n-1}} \geqslant \frac{a}{2C} n^\varepsilon, \qquad \text{for all sufficiently large} \quad n \in L',$$

becomes unbounded for $n \in L'$, which contradicts our principal assumption (3). Hence, Lemma 5 is proved.

The proof of Lemma 5 yields that p_n^1 has exact degree n for all $n \in L$.

For $X_n^1 = \{x'_1 < \cdots < x'_{n+1}\}$ we now compare the products

$$\prod_{\substack{\nu=1\\|\nu-j|\geqslant 2}}^{n+1} |x'_j - x'_\nu| \qquad \text{to the products} \qquad \prod_{\substack{\nu=1\\|\nu-j|\geqslant 2}}^{n+1} |x'_j - \xi'_\nu|.$$

We recall that $\xi'_\nu = \xi'_\nu(n, 1)$ denotes the left end point of the sign component of E_n that contains x'_ν, $1 \leqslant \nu \leqslant n+1$. In geometric average over $3 \leqslant j \leqslant n-1$ the first product is no larger than the second multiplied by a constant factor, as $n \in L$ increases:

Lemma 6. *Let $X_n^1 = \{x_j, j \in I_1\} = \{x'_1 < \cdots < x'_{n+1}\}$ be the solution of $B(n, 1)$, $n \in L$. Then there exists a constant $C_6 > 0$, not depending on $n \in L$, such that*

$$\left(\prod_{j=3}^{n-1} \left\{ \prod_{\substack{\nu=1\\|\nu-j|\geqslant 2}}^{n+1} |x'_j - x'_\nu| \right\} \left\{ \prod_{\substack{\nu=1\\|\nu-j|\geqslant 2}}^{n+1} |x'_j - \xi'_\nu| \right\}^{-1} \right)^{1/n} \leqslant C_6.$$

Proof of Lemma 6. 1. For the solutions $X_n^1 = \{x'_1 < \cdots < x'_{n+1}\}$ of $B(n, 1)$, $n \in L$, we put

$$\beta_\nu = \beta_\nu(n) := |x'_\nu - \xi'_\nu|, \qquad \text{for all} \quad 1 \leqslant \nu \leqslant n+1.$$

Lemma 1 yields $\beta_\nu \leqslant C_1 d_n(\nu)$ for all $n \in L$, $1 \leqslant \nu \leqslant n+1$.

Then, for every $n \in L$ and $3 \leqslant j \leqslant n-1$, we have

$$\begin{aligned}
&\left\{ \prod_{\substack{\nu=1\\|\nu-j|\geqslant 2}}^{n+1} |x'_j - x'_\nu| \right\} \left\{ \prod_{\substack{\nu=1\\|\nu-j|\geqslant 2}}^{n+1} |x'_j - \xi'_\nu| \right\}^{-1} \\
&\quad = \prod_{\nu=1}^{j-2} \left(1 - \frac{|x'_\nu - \xi'_\nu|}{|x'_j - \xi'_\nu|} \right) \prod_{\nu=j+2}^{n+1} \left(1 + \frac{|x'_\nu - \xi'_\nu|}{|x'_j - \xi'_\nu|} \right) \\
&\quad \leqslant \exp \left\{ -\sum_{\nu=1}^{j-2} \frac{\beta_\nu}{|x'_j - \xi'_\nu|} + \sum_{\nu=j+2}^{n+1} \frac{\beta_\nu}{|x'_j - \xi'_\nu|} \right\}. \qquad (5)
\end{aligned}$$

2. Since $|p_n^1(x'_1)| = |p_n^1(x'_2)| = 1 < |p_n^1(y)|$, $y = (\eta_1 + \xi_2)/2$, there must be a local extremum of p_n^1 in (x'_1, x'_2). Therefore, we obtain that the zeros $\zeta_0, \zeta_2, ..., \zeta_n$ of p_n^1 are arranged in the following way:

$$\zeta_1 := \zeta_0 < x'_1 < x'_2 < \zeta_2 < x'_3 < \cdots < x'_n < \zeta_n < x'_{n+1}.$$

For convenience of notation we put $\zeta_1 := \zeta_0$.

The crucial step of the proof will be to replace the sums occuring in the exponential term (5) by sums involving the zeros ζ_j.

There exists some $C>0$, not depending on $n\in L$ or $3\leqslant j\leqslant n-1$, such that the following estimates hold

$$\left|\sum_{\nu=1}^{j-2}\frac{\beta_\nu}{|x'_j-\xi'_\nu|}-\sum_{\nu=2}^{j-1}\frac{\beta_\nu}{|\zeta_j-x'_\nu|}\right|\leqslant C,$$

$$\left|\sum_{\nu=j+2}^{n+1}\frac{\beta_\nu}{|x'_j-\xi'_\nu|}-\sum_{\nu=j+1}^{n}\frac{\beta_\nu}{|\zeta_j-x'_\nu|}\right|\leqslant C.$$

We give the computation only for the first difference. By Lemma 1 we have

$$\left|\sum_{\nu=1}^{j-2}\frac{\beta_\nu}{|x'_j-\xi'_\nu|}-\sum_{\nu=2}^{j-1}\frac{\beta_\nu}{|\zeta_j-x'_\nu|}\right|$$

$$\leqslant\frac{\beta_1}{|x'_j-\xi'_1|}+\frac{\beta_{j-1}}{|\zeta_j-x'_{j-1}|}+\sum_{\nu=2}^{j-2}\frac{\beta_\nu(|x'_\nu-\xi'_\nu|+|x'_j-\zeta_j|)}{|(x'_j-\xi'_\nu)(\zeta_j-x'_\nu)|}$$

$$\leqslant\frac{C_1\,d_n(1)}{D_1\,d_n(j,1)}+\frac{C_1\,d_n(j-1)}{D_1\,d_n(j,j-1)}+\frac{C_1^2}{D_1^2}\sum_{\nu=2}^{j-2}\frac{d_n(\nu)(d_n(\nu)+d_n(j))}{d_n(j,\nu)\,d_n(j,\nu)}.$$

Lemma 2(d) yields that the difference may be estimated by some $C>0$.

3. Replacing the sums in (5) according to the second part of the proof we get

$$\prod_{j=3}^{n-1}\left\{\prod_{\substack{\nu=1\\|\nu-j|\geqslant 2}}^{n+1}|x'_j-x'_\nu|\right\}\left\{\prod_{\substack{\nu=1\\|\nu-j|\geqslant 2}}^{n+1}|x'_j-\xi'_\nu|\right\}^{-1}$$

$$\leqslant\exp\left\{2C(n-3)+\sum_{j=3}^{n-1}\left(-\sum_{\nu=2}^{j-1}\frac{\beta_\nu}{|\zeta_j-x'_\nu|}+\sum_{\nu=j+1}^{n}\frac{\beta_\nu}{|\zeta_j-x'_\nu|}\right)\right\}.$$

Further, we have $x'_\nu-\zeta_j<0$, $\nu<j$, and $x'_\nu-\zeta_j>0$, $\nu>j$. For the sum occuring in the exponential term above we therefore get

$$\left|\sum_{j=3}^{n-1}\left(-\sum_{\nu=2}^{j-1}\frac{\beta_\nu}{|\zeta_j-x'_\nu|}+\sum_{\nu=j+1}^{n}\frac{\beta_\nu}{|\zeta_j-x'_\nu|}\right)\right|$$

$$=\left|\sum_{\nu=2}^{n-2}\beta_\nu\sum_{j=\nu+1}^{n-1}\frac{1}{|x'_\nu-\zeta_j|}-\sum_{\nu=4}^{n}\beta_\nu\sum_{j=3}^{\nu-1}\frac{1}{|x'_\nu-\zeta_j|}\right|$$

$$\leqslant\sum_{\nu=4}^{n-2}\left|\beta_\nu\left(\sum_{j=\nu+1}^{n-1}\frac{1}{x'_\nu-\zeta_j}+\sum_{j=3}^{\nu-1}\frac{1}{x'_\nu-\zeta_j}\right)\right|$$

$$+\sum_{\nu=2}^{3}\beta_\nu\sum_{j=\nu+1}^{n-1}\frac{1}{|x'_\nu-\zeta_j|}+\sum_{\nu=n-1}^{n}\beta_\nu\sum_{j=3}^{\nu-1}\frac{1}{|x'_\nu-\zeta_j|}.$$

Lemma 1 and Lemma 2(b) yield that

$$\begin{aligned}
&\left|\sum_{j=3}^{n-1}\left(-\sum_{v=2}^{j-1}\frac{\beta_v}{|\zeta_j-x'_v|}+\sum_{v=j+1}^{n}\frac{\beta_v}{|\zeta_j-x'_v|}\right)\right| \\
&\quad\leqslant \sum_{v=4}^{n-2}\left|\beta_v\left(\sum_{j=v+1}^{n-1}\frac{1}{x'_v-\zeta_j}+\sum_{j=3}^{v-1}\frac{1}{x'_v-\zeta_j}\right)\right| \\
&\qquad+\frac{C_1}{D_1}\sum_{v=2}^{3}\sum_{j=v+1}^{n-1}\frac{d_n(v)}{d_n(v,j)}+\sum_{v=n-1}^{n}\sum_{j=3}^{v-1}\frac{d_n(v)}{d_n(v,j)} \\
&\quad\leqslant \sum_{v=4}^{n-2}\left|\beta_v\left(\sum_{j=v+1}^{n-1}\frac{1}{x'_v-\zeta_j}+\sum_{j=3}^{v-1}\frac{1}{x'_v-\zeta_j}\right)\right|+4\frac{C_1}{D_1}C_2\log(n).
\end{aligned}$$

Since $|x'_v-\zeta_1|\geqslant|x'_v-\zeta_2|$ and by Lemma 1 and Lemma 2(a), we get

$$\begin{aligned}
&\left|\sum_{j=3}^{n-1}\left(-\sum_{v=2}^{j-1}\frac{\beta_v}{|\zeta_j-x'_v|}+\sum_{v=j+1}^{n}\frac{\beta_v}{|\zeta_j-x'_v|}\right)\right| \\
&\quad\leqslant \sum_{v=4}^{n-2}\left|\beta_v\sum_{j=1}^{n}\frac{1}{x'_v-\zeta_j}\right| \\
&\qquad+\sum_{v=4}^{n-2}\beta_v\left(\frac{1}{|x'_v-\zeta_n|}+\frac{1}{|x'_v-\zeta_v|}+\frac{1}{|x'_v-\zeta_2|}+\frac{1}{|x'_v-\zeta_1|}\right) \\
&\qquad+4\frac{C_1}{D_1}C_2\log(n) \\
&\quad\leqslant \sum_{v=4}^{n-2}\beta_v\left|\sum_{j=1}^{n}\frac{1}{x'_v-\zeta_j}\right| \\
&\qquad+\frac{C_1}{D_1}\sum_{v=4}^{n-2}d_n(v)\left(\frac{1}{d_n(v,n)}+\frac{1}{d_n(v)}+\frac{1}{d_n(v,2)}+\frac{1}{d_n(v,2)}\right) \\
&\qquad+4\frac{C_1}{D_1}C_2\log(n) \\
&\quad\leqslant \sum_{v=4}^{n-2}\beta_v\left|\sum_{j=1}^{n}\frac{1}{x'_v-\zeta_j}\right| \\
&\qquad+\frac{C_1}{D_1}(C_2\log(n)+(n-5)+C_2\log(n)+C_2\log(n)+4C_2\log(n)) \\
&\quad\leqslant \sum_{v=4}^{n-2}\beta_v\left|\sum_{j=1}^{n}\frac{1}{x'_v-\zeta_j}\right|+Cn,
\end{aligned}$$

for some $C>0$, not depending on $n\in L$.

Since $|p_n^1(x'_\nu)|=1$, we can now estimate the remaining sum by (4) and Lemma 2(c)

$$\sum_{\nu=4}^{n-2}\beta_\nu\left|\sum_{j=1}^{n}\frac{1}{x'_\nu-\zeta_j}\right|=\sum_{\nu=4}^{n-1}\beta_\nu\left|\frac{(p_n^1)'(x'_\nu)}{p_n^1(x'_\nu)}\right|\leqslant\sum_{\nu=4}^{n-2}\beta_\nu\frac{Mn}{\sqrt{1-(x'_\nu)^2}}$$

$$\leqslant C_1\frac{Mn}{D_1}\left(\sum_{\nu=4}^{n-2}\frac{d_n(\nu)}{\sqrt{d_n(\nu,n+2)\,d_n(\nu,1)}}\right)\leqslant Cn,$$

for some $C>0$, not depending on $n\in L$.

This implies the estimate of Lemma 6 with some suitable constant C_6.

Lemma 6 yields

LEMMA 7. *Suppose that $\varepsilon>0$. For the solutions $X_n^1=\{x_j,\ j\in I_1\}=\{x'_1<\cdots<x'_{n+1}\}$ of $B(n,1)$, $n\in L$, let $b_n(\varepsilon)$ denote the number of indices $3\leqslant j\leqslant n-1$ such that*

$$\prod_{\substack{\nu=1\\|\nu-j|\geqslant 2}}^{n+1}|x'_j-\xi'_\nu|\geqslant\frac{1}{n^\varepsilon}\prod_{\substack{\nu=1\\|\nu-j|\geqslant 2}}^{n+1}|x'_j-x'_\nu|.$$

Then we have

$$\liminf_{n\in L}\frac{b_n(\varepsilon)}{n}>0.$$

Proof of Lemma 7. For every $n\in L$ there exist at least $n-3-b_n(\varepsilon)$ indices $3\leqslant j\leqslant n-1$ such that

$$\prod_{\substack{\nu=1\\|\nu-j|\geqslant 2}}^{n+1}|x'_j-x'_\nu|\geqslant n^\varepsilon\prod_{\substack{\nu=1\\|\nu-j|\geqslant 2}}^{n+1}|x'_j-\xi'_\nu|.$$

Since $|x'_j-x'_\nu|\geqslant|x'_j-\xi'_\nu|$ for all $\nu\geqslant j+2$ and $|x'_j-x'_{\nu-1}|\geqslant|x'_j-\xi'_\nu|$ for all $\nu\leqslant j-2$, we obtain that with some $D>0$

$$\prod_{\substack{\nu=1\\|\nu-j|\geqslant 2}}^{n+1}|x'_j-x'_\nu|\geqslant\prod_{\nu=1}^{j-2}|x'_j-x'_\nu|\prod_{\nu=j+2}^{n+1}|x'_j-\xi'_\nu|$$

$$\geqslant|x'_j-x'_{j-2}|\prod_{\nu=2}^{j-2}|x'_j-\xi'_\nu|\prod_{\nu=j+2}^{n+1}|x'_j-\xi'_\nu|$$

$$\geqslant \frac{|x'_j - x'_{j-2}|}{|x'_j - \xi'_1|} \prod_{\substack{\nu=1 \\ |\nu - j| \geqslant 2}}^{n+1} |x'_j - \xi'_\nu|$$

$$\geqslant \frac{D_1}{2} d_n(j) \prod_{\substack{\nu=1 \\ |\nu - j| \geqslant 2}}^{n+1} |x'_j - \xi'_\nu|$$

$$\geqslant D \frac{1}{n^2} \prod_{\substack{\nu=1 \\ |\nu - j| \geqslant 2}}^{n+1} |x'_j - \xi'_\nu|,$$

for all $n \in L$ and all $3 \leqslant j \leqslant n-1$.

Using these two estimates and Lemma 6 we get

$$C_6 \geqslant \prod_{j=3}^{n-1} \left(\left\{ \prod_{\substack{\nu=1 \\ |\nu - j| \geqslant 2}}^{n+1} |x'_j - x'_\nu| \right\} \left\{ \prod_{\substack{\nu=1 \\ |\nu - j| \geqslant 2}}^{n+1} |x'_j - \xi'_\nu| \right\}^{-1} \right)^{1/n}$$

$$\geqslant (n^\varepsilon)^{(n-3-b_n(\varepsilon))/n} \left(\frac{D}{n^2} \right)^{b_n(\varepsilon)/n}, \qquad \text{for all} \quad n \in L,$$

which leads to a contradiction if we assume that $\liminf_{n \in L}(b_n(\varepsilon)/n) = 0$. Hence, Lemma 7 is proved.

Next, we define a special sequence of indices $k^* = k^*(n)$, $n \in L$.

DEFINITION. For $n \in L$ we choose $k^* = k^*(n) \in \{3, ..., n\}$ such that

$$d_n(k^*)^3 \prod_{\substack{\nu=1 \\ |\nu - k^*| \geqslant 2}}^{n+2} |\xi_{k^*} - \xi_\nu| = \max_{3 \leqslant j \leqslant n} d_n(j)^3 \prod_{\substack{\nu=1 \\ |\nu - j| \geqslant 2}}^{n+2} |\xi_j - \xi_\nu|, \tag{6}$$

where, for every $n \in L$, the $\xi_1, ..., \xi_{n+2}$ denote the left end points of $E_n^1, ..., E_n^{n+2}$.

LEMMA 8. *Let $k^* = k^*(n)$, $n \in L$, be defined by* (6).

(a) *Then, for every $\varepsilon > 0$, we have*

$$\liminf_{n \in L} n^\varepsilon 2^n d_n(k^*)^3 \prod_{\substack{\nu=1 \\ |\nu - k^*| \geqslant 2}}^{n+2} |\xi_{k^*} - \xi_\nu| > 0.$$

(b) *With $D_4 > 0$ from Lemma* 4 *it follows that for $k^* = k^*(n)$*

$$|\xi_{k^*} - \eta_{k^*-1}|, |\xi_{k^*+1} - \eta_{k^*}| \geqslant D_4 d_n(k^*), \qquad \textit{for all} \quad n \in L. \tag{7}$$

Proof of Lemma 8. Let $\varepsilon > 0$ be given.

1. We first prove a corresponding estimate for a suitable sequence of points $x_k \in X_n^1$, where $k = k(n)$, $n \in L$.

For the solutions $X_n^1 = \{x_j, j \in I_1\} = \{x'_1 < \cdots < x'_{n+1}\}$ of $B(n, 1)$, $n \in L$, we have by Lemma 5 and Lemma 7

$$\lim_{n \in L} a_n(\varepsilon/4)/n = 0 \qquad \text{and} \qquad b(\varepsilon) := \liminf_{n \in L} b_n(\varepsilon/4)/n > 0.$$

We recall that for every $n \in L$ we denote by ξ_{j_1} the left end point of $E_n^{j_1}$, where $j_1 = j_1(n)$ and $I_1 = I_1(n) = \{1, ..., n+2\} \setminus \{j_1\}$.

Thus, for each $n \in L$ sufficiently large, there exist at least $b(\varepsilon)\, n/2$ indices l among the indices $3 \leqslant l \leqslant n-1$ such that

$$\prod_{\substack{\nu=1 \\ \nu \neq l}}^{n+1} |x'_l - x'_\nu| \geqslant \frac{1}{n^{\varepsilon/4}} \frac{1}{d_n(l)} \frac{1}{2^n},$$

and

$$\prod_{\substack{\nu=1 \\ |\nu - l| \geqslant 2}}^{n+1} |x'_l - \xi'_\nu| \geqslant \frac{1}{n^{\varepsilon/4}} \prod_{\substack{\nu=1 \\ |\nu - l| \geqslant 2}}^{n+1} |x'_l - x'_\nu|.$$

For all sufficiently large $n \in L$ we may therefore choose such an index $l = l(n)$ such that $|l(n) - j_1(n)| \geqslant b(\varepsilon)\, n/4$. Using Lemma 1, we obtain that

$$|x'_l - \xi_{j_1}| > D(\varepsilon), \qquad \text{for all} \quad n \in L,$$

for some suitable $D(\varepsilon) > 0$, depending on ε but not on $n \in L$.

For every $n \in L$ we now define $k = k(n)$, $n \in L$, by $x_{k(n)} = x'_{l(n)} \in X_n^1$. It follows that $k(n) = l(n)$ if $j_1(n) > k(n)$ and $k(n) = l(n) + 1$ if $j_1(n) < k(n)$. In particular, $3 \leqslant k(n) \leqslant n$ for all $n \in L$ and $\lim_{n \in L} |k(n) - j_1(n)| = \infty$. Lemma 1 yields that for $k = k(n)$ and $l = l(n)$

$$\begin{aligned}
\prod_{\substack{\nu=1 \\ |\nu - k| \geqslant 2}}^{n+2} |x_k - \xi_\nu| &= |x'_l - \xi_{j_1}| \prod_{\substack{\nu=1 \\ |\nu - l| \geqslant 2}}^{n+1} |x'_l - \xi'_\nu| \\
&\geqslant D(\varepsilon) \frac{1}{n^{\varepsilon/4}} \prod_{\substack{\nu=1 \\ |\nu - l| \geqslant 2}}^{n+1} |x'_l - x'_\nu| \\
&\geqslant D(\varepsilon) \frac{1}{n^{\varepsilon/4}} \frac{1}{|(x'_l - x'_{l-1})(x'_l - x'_{l+1})|} \prod_{\substack{\nu=1 \\ \nu \neq l}}^{n+1} |x'_l - x'_\nu| \\
&\geqslant \frac{D(\varepsilon)}{C_1^2} \frac{1}{n^{\varepsilon/2}} \frac{1}{d_n(l)^3} \frac{1}{2^n}, \qquad \text{for all sufficiently large} \quad n \in L.
\end{aligned}$$

Since $|k(n)-l(n)| \leqslant 1$, we get that for some suitable $D(\varepsilon)>0$

$$\prod_{\substack{\nu=1\\|\nu-k|\geqslant 2}}^{n+2} |x_k-\xi_\nu| \geqslant \frac{D(\varepsilon)}{n^{\varepsilon/2}} \frac{1}{d_n(k)^3} \frac{1}{2^n}, \qquad \text{for all sufficiently large} \quad n\in L. \tag{8}$$

2. Next, we show that the points $x_k \in X_n^1$ in (8) may be replaced by the corresponding $\xi_k \in E_n^k$, where $k=k(n)$, $n\in L$.

For the sequence $k=k(n)$, $n\in L$, defined in part 1, let p_n^k and X_n^k denote the solution of $B(n, k)$. We use the notation $X_n^k=\{y_j,\ j\in I_k\}=\{y'_1<\cdots<y'_{n+1}\}$ to avoid confusion with the solutions $X_n^1=\{x_j,\ j\in I_1\}=\{x'_1<\cdots<x'_{n+1}\}$ of $B(n, 1)$.

The polynomial p_n^k has exactly $n-1$ zeros $\zeta_1, ..., \zeta_{k'-1}, \zeta_{k'+1}, ..., \zeta_n$ in $[y'_1, y'_{n+1}]$ which are ordered in the following way

$$y'_1<\zeta_1<y'_2<\cdots<\zeta_{k'-1}<y'_{k'}:=y_k=\xi_k<\xi_{k+1}$$

$$\xi_{k+1}=y_{k+1}=:y'_{k'+1}<\zeta_{k'+1}<\cdots<y'_n<\zeta_n<y'_{n+1},$$

and there may exist one additional zero $\zeta_0\notin[y'_1, y'_{n+1}]$ of p_n^k.

We have $\max_{E_n^k} \sigma_n(x)\, p_n^k(x)=-1$, which implies that

$$1=|p_n^k(\xi_k)|\leqslant|p_n^k(x)|\leqslant M, \qquad \text{for all} \quad x\in E_n^k \quad \text{and} \quad k=k(n), \quad n\in L.$$

Therefore, there exists some $D>0$ such that

$$\prod_{\substack{\nu=1\\ \nu\neq k'}}^{n} \frac{|\xi_k-\zeta_\nu|}{|x_k-\zeta_\nu|} \geqslant \frac{1}{M}\left\{\frac{|x_k-\zeta_0|}{|\xi_k-\zeta_0|}\right\}\geqslant D, \qquad \text{for all} \quad n\in L. \tag{9}$$

Here, the factor in $\{\cdots\}$ appears only if p_n^k has exact degree n. Since $3\leqslant k=k(n)\leqslant n$, it is not difficult to check that $\{\cdots\}$ is bounded away from 0 by some positive constant, as $n\in L$ increases.

We shall prove that there exists some $C>0$ such that for $k=k(n)$

$$\prod_{\substack{\nu=1\\|\nu-k|\geqslant 2}}^{n+2} \frac{|x_k-\xi_\nu|}{|\xi_k-\xi_\nu|} \prod_{\substack{\nu=1\\ \nu\neq k'}}^{n} \frac{|\xi_k-\zeta_\nu|}{|x_k-\zeta_\nu|}\leqslant C, \qquad \text{for all} \quad n\in L. \tag{10}$$

For the sake of shortness we will only consider the case $j_k(n)>k(n)$, i.e., $k=k'$:

$$\prod_{\substack{\nu=1\\|\nu-k|\geqslant 2}}^{n+2} \frac{|x_k-\xi_\nu|}{|\xi_k-\xi_\nu|} \prod_{\substack{\nu=1\\ \nu\neq k}}^{n} \frac{|\xi_k-\zeta_\nu|}{|x_k-\zeta_\nu|}$$

$$= \prod_{\nu=1}^{k-2}\left(1+\frac{|x_k-\xi_k|}{|\xi_k-\xi_\nu|}\right)\prod_{\nu=1}^{k-1}\left(1-\frac{|x_k-\xi_k|}{|x_k-\zeta_\nu|}\right)$$

$$\times \prod_{\nu=k+2}^{n+2}\left(1-\frac{|x_k-\xi_k|}{|\xi_k-\xi_\nu|}\right)\prod_{\nu=k+1}^{n}\left(1+\frac{|x_k-\xi_k|}{|x_k-\zeta_\nu|}\right)$$

$$\leqslant \exp\left\{\sum_{\nu=1}^{k-2}\frac{|x_k-\xi_k|\,(|x_k-\zeta_\nu|-|\xi_k-\xi_\nu|)}{|(\xi_k-\xi_\nu)(x_k-\zeta_\nu)|}\right.$$

$$+\sum_{\nu=k+2}^{n}\frac{|x_k-\xi_k|\,(|\xi_k-\xi_\nu|-|x_k-\zeta_\nu|)}{|(\xi_k-\xi_\nu)(x_k-\zeta_\nu)|}$$

$$\left.-\frac{|x_k-\xi_k|}{|x_k-\zeta_{k-1}|}+\frac{|x_k-\xi_k|}{|x_k-\zeta_{k+1}|}-\frac{|x_k-\xi_k|}{|\xi_k-\xi_{n+1}|}-\frac{|x_k-\xi_k|}{|\xi_k-\xi_{n+2}|}\right\}$$

$$\leqslant \exp\left\{\sum_{\nu=1}^{k-2}\frac{|x_k-\xi_k|\,(|x_k-\xi_k|+|\zeta_\nu-\xi_\nu|)}{|(\xi_k-\xi_\nu)(x_k-\zeta_\nu)|}\right.$$

$$\left.+\sum_{\nu=k+2}^{n}\frac{|x_k-\xi_k|\,(|x_k-\xi_k|+|\zeta_\nu-\xi_\nu|)}{|(\xi_k-\xi_\nu)(x_k-\zeta_\nu)|}+\frac{|x_k-\xi_k|}{|x_k-\zeta_{k+1}|}\right\}.$$

The crucial part here is to obtain control from below for the distances $|\xi_k-\xi_\nu|$ appearing in the denominators.

Since $\lim_{n\in L}|k(n)-j_1(n)|=\infty$, we have $\{k(n)-2, k(n)-1, k(n), k(n)+1\} \subset I_1(n)$ for all sufficiently large $n\in L$. Thus, there exists a point of X_n^1 and a zero of p_n^1 in $[\xi_{k-2}, \xi_k]\supset[x_{k-2}, x_{k-1}]$ and in $[\xi_k, \xi_{k+2}]\supset[x_k, x_{k+1}]$.

Lemma 1 and Lemma 2(d) yield that for some suitable $C>0$

$$\leqslant \exp\left\{\frac{2C_1^2}{D_1^2}\sum_{\substack{\nu=1\\ \nu\neq k}}^{n+2}\frac{d_n(k)(d_n(k)+d_n(\nu))}{d_n(k,\nu)^2}+\frac{C_1}{D_1}\frac{d_n(k)}{d_n(k,k+1)}\right\}$$

$$\leqslant C, \qquad \text{for all sufficiently large} \quad n\in L.$$

We note that the case $j_k(n)>k(n)$ is somewhat more inconvenient to write down but may be treated in exactly the same manner.

With some suitable $C>0$ we have by (9), (10) and (8)

$$\prod_{\substack{\nu=1\\|\nu-k|\geqslant 2}}^{n+2} |\xi_k-\xi_\nu| \geqslant \frac{1}{C} \prod_{\substack{\nu=1\\|\nu-k|\geqslant 2}}^{n+2} |x_k-\xi_\nu|$$

$$\geqslant \frac{D(\varepsilon)}{C}\frac{1}{n^{\varepsilon/2}}\frac{1}{d_n(k)^3}\frac{1}{2^n}, \qquad \text{for all sufficiently large} \quad n\in L.$$

By the definition of $k^*=k^*(n)$, $n\in L$, this implies the first part of Lemma 8.

The second part then follows immediately from Lemma 4, and Lemma 8 is proven.

We give some further properties of the sequence $k^*=k^*(n)$, $n\in L$.

LEMMA 9. *For the solution* $X_n^{k^*}=\{x_j,\ j\in I_{k^*}\}$ *of* $B(n,k^*)$ *let* $I_{k^*}=\{1,...,n+2\}\backslash\{j_{k^*}\}$, *where* $k^*=k^*(n)$, $j_{k^*}=j_{k^*}(n)$, $n\in L$. *Then we have*

$$\lim_{n\in L} k^*(n)=\infty \qquad \text{and} \qquad \lim_{n\in L} |k^*(n)-j_{k^*}(n)|=\infty.$$

Proof of Lemma 9. 1. Let us assume that there exists a subsequence L' of L and some constant $m\in\mathbb{N}$ such that

$$k^*(n)\leqslant m \qquad \text{for all} \quad n\in L'.$$

For the solutions $p_n^{k^*}$ and $X_n^{k^*}=\{x_j,\ j\in I_{k^*}\}=\{x_1'<\cdots<x_{n+1}'\}$ of $B(n,k^*)$, $k^*=k^*(n)$, $n\in L'$, we then have by Lemma 1 and (7) in Lemma 8

$$\prod_{\substack{j\in I_{k^*}\\ j\neq k^*}} |x_{k^*}-x_j| \geqslant |x_{k^*}-x_{k^*+1}| \prod_{\substack{j\in I_{k^*}\\ j<k^*}} |x_{k^*}-x_j| \prod_{\substack{j\in I_{k^*}\\ j>k^*+1}} |x_{k^*}-\xi_j|$$

$$=|x_{k^*}-x_{k^*+1}| \prod_{\substack{j\in I_{k^*}\\ j<k^*}} \frac{|x_{k^*}-x_j|}{|\xi_{k^*}-\xi_j|} \prod_{\substack{j\in I_{k^*}\\ j\neq k^*,\,k^*+1}} |x_{k^*}-\xi_j|$$

$$\geqslant |x_{k^*}-x_{k^*+1}| \left(\frac{D_1}{C_1}\right)^m \prod_{\substack{j\in I_{k^*}\\ j\neq k^*,\,k^*+1}} |x_{k^*}-\xi_j|$$

$$=|x_{k^*}-x_{k^*+1}| \left(\frac{D_1}{C_1}\right)^m \frac{|x_{k^*}-\xi_{k^*-1}|}{|x_{k^*}-\xi_{j_{k^*}}|} \prod_{\substack{j=1\\|j-k^*|\geqslant 2}}^{n+2} |x_{k^*}-\xi_j|$$

$$\geqslant |x_{k^*} - x_{k^*+1}| \left(\frac{D_1}{C_1}\right)^m \frac{D_4 d_n(k^*)}{|x_{k^*} - \xi_{j_{k^*}}|} \prod_{\substack{j=1 \\ |j-k^*| \geqslant 2}}^{n+2} |x_{k^*} - \xi_j|$$

$$= |x_{k^*} - x_{k^*+1}| \left(\frac{D_1}{C_1}\right)^m \frac{D_4 d_n(k^*)}{2} \prod_{\substack{j=1 \\ |j-k^*| \geqslant 2}}^{n+2} |\xi_{k^*} - \xi_j|,$$

since $x_{k^*} = \xi_{k^*}$ according to Theorem 3. Choosing $\delta > 0$ from Lemma 3 and $\varepsilon = \delta/2$ in Lemma 8, this contradicts Lemma 3.

2. Let us assume that there exists a subsequence L' of L and some constant $m \in \mathbb{N}$ such that

$$|k^*(n) - j_{k^*}(n)| \leqslant m \qquad \text{for all} \quad n \in L'.$$

Then it follows that for some $C > 0$ we have

$$|\xi_{k^*} - \eta_{j_{k^*}}| \leqslant C_1\, d_n(k^*, j_{k^*}) \leqslant C d_n(k^*), \qquad \text{for all} \quad n \in L'.$$

For the sake of shortness we will only consider the case $j_{k^*}(n) < k^*(n)$. Since $\xi_{k^*} = x_{k^*}$ and $\xi_{k^*+1} = x_{k^*+1}$, Lemma 3 and (7) yield

$$\prod_{j=1}^{k^*-2} |\xi_{k^*} - \eta_j| \prod_{j=k^*+2}^{n+2} |\xi_{k^*} - \xi_j|$$

$$= \frac{|\xi_{k^*} - \eta_{j_{k^*}}|}{|\xi_{k^*} - \eta_{k^*-1}|} \frac{1}{|\xi_{k^*} - \xi_{k^*+1}|} \prod_{\substack{j \in I_{k^*} \\ j < k^*}} |\xi_{k^*} - \eta_j| \prod_{\substack{j \in I_{k^*} \\ j > k^*}} |\xi_{k^*} - \xi_j|$$

$$\leqslant \frac{|\xi_{k^*} - \eta_{j_{k^*}}|}{|\xi_{k^*} - \eta_{k^*-1}|} \frac{1}{|\xi_{k^*} - \xi_{k^*+1}|} \prod_{\substack{j \in I_{k^*} \\ j \neq k^*}} |\xi_{k^*} - x_j|$$

$$= \frac{|\xi_{k^*} - \eta_{j_{k^*}}|}{|\xi_{k^*} - \eta_{k^*-1}|} \frac{1}{|x_{k^*} - x_{k^*+1}|} \prod_{\substack{j \in I_{k^*} \\ j \neq k^*}} |x_{k^*} - x_j|$$

$$\leqslant \frac{C d_n(k^*)}{D_4\, d_n(k^*)} \frac{C_3}{n^\delta} \frac{1}{d_n(k^*)^2} \frac{1}{2^n}$$

$$\leqslant C \frac{1}{n^\delta} \frac{1}{d_n(k^*)^2} \frac{1}{2^n}, \qquad \text{for all sufficiently large} \quad n \in L',$$

for some suitable $C > 0$.

On the other hand, we have

$$\prod_{j=1}^{k^*-2} |\xi_{k^*}-\eta_j| \prod_{j=k^*+2}^{n+2} |\xi_{k^*}-\xi_j|$$
$$\geqslant \prod_{j=2}^{k^*-1} |\xi_{k^*}-\xi_j| \prod_{j=k^*+2}^{n+2} |\xi_{k^*}-\xi_j|$$
$$=\frac{|\xi_{k^*}-\xi_{k^*-1}|}{|\xi_{k^*}-\xi_1|} \prod_{\substack{j=1\\|j-k^*|\geqslant 2}}^{n+2} |\xi_{k^*}-\xi_j|$$
$$\geqslant \frac{D_4\, d_n(k^*)}{2} \prod_{\substack{j=1\\|j-k^*|\geqslant 2}}^{n+2} |\xi_{k^*}-\xi_j|, \qquad \text{for all sufficiently large} \quad n\in L'.$$

Choosing $\varepsilon=\delta/2$ in Lemma 8 we obtain a contradiction.
Hence, Lemma 9 is proved.

To complete the proof of our main Theorem 1 let $p_n^{k^*}$ and $X_n^{k^*}=\{x_j,\ j\in I_{k^*}\}$ $=\{x'_1<\cdots<x'_{n+1}\}$ denote the solutions of $B(n, k^*)$, where $k^*=k^*(n), n\in L$, is defined in (6).

We recall that $I_{k^*}=I_{k^*}(n)=\{1, ..., n+2\}\backslash\{j_{k^*}\}$, where $j_{k^*}=j_{k^*}(n)$, $n\in L$.
If $j_{k^*}(n)>k^*(n)$, we put $l=l(n):=3$. If $j_{k^*}(n)<k^*(n)$, we choose $l=l(n)\in\{j_{k^*}(n)+1, ..., k^*(n)-1\}$ as small as possible such that

$$\frac{|\xi_j-\xi_{j_{k^*}}|}{|\xi_{k^*}-\xi_{j_{k^*}}|}\geqslant\frac{1}{2} \qquad \text{for all} \quad l(n)<j<k^*(n).$$

By Lemma 9 it is easy to check that

$$\lim_{n\in L} k^*(n)-l(n)=\infty.$$

Further, we have

$$\max_{E_n^j} \sigma_n(x)\, p_n^{k^*}(x)=-1 \qquad \text{for all} \quad l(n)\leqslant j\leqslant k^*(n),$$

and therefore

$$\min_{E_n^j} |p_n^{k^*}(x)|=1 \qquad \text{for all} \quad l(n)\leqslant j\leqslant k^*(n).$$

We consider $|p_n^{k^*}(y)|$ at the point $y=y(n)=(\eta_{k^*}+\xi_{k^*+1})/2$, $n\in L$.

For every $n\in L$ let us define points $y_j=y_j(n)\in E_n^j$, $j\in I_{k^*}(n)$, by

$$y_j=y_j(n):=\xi_j, \qquad \text{if} \quad l(n)\leqslant j\leqslant k^*(n),$$

and

$$y_j = y_j(n) := x_j, \qquad \text{for all other} \quad j \in I_{k^*}(n).$$

By the sign structure of $p_n^{k^*}(y_j)$ and since $|p_n^{k^*}(y_j)| \geqslant 1$, the Lagrange interpolation formula yields

$$\begin{aligned} |p_n^{k^*}(y)| &= \left(\prod_{\nu \in I_{k^*}} |y - y_\nu| \right) \sum_{j \in I_{k^*}} \frac{|p_n^{k^*}(y_j)|}{\prod_{\nu \in I_{k^*},\, \nu \neq j} |y_j - y_\nu|} \frac{1}{|y_j - y|} \\ &\geqslant \left(\prod_{\nu \in I_{k^*}} |y - y_\nu| \right) \sum_{l < j < k^*} \frac{1}{\prod_{\nu \in I_{k^*},\, \nu \neq j} |y_j - y_\nu|} \frac{1}{|y_j - y|}. \end{aligned}$$

1. Since $1 = |p_n^{k^*}(\xi_{k^*})| \leqslant |p_n^{k^*}(y)| \leqslant M$, it follows similarly to the second part of the proof of Lemma 8 that for some $D > 0$

$$\prod_{\substack{\nu \in I_{k^*} \\ \nu \neq k^*}} |y - y_\nu| \geqslant D \prod_{\substack{\nu \in I_{k^*} \\ \nu \neq k^*}} |\xi_{k^*} - y_\nu|, \qquad \text{for all sufficiently large} \quad n \in L.$$

We skip the computation and just note that

$$\left(\prod_{\substack{\nu \in I_{k^*} \\ \nu \neq k^*}} \frac{|\xi_{k^*} - y_\nu|}{|y - y_\nu|} \right) \left| \frac{p_n^{k^*}(y)}{p_n^{k^*}(\xi_{k^*})} \right|$$

remains bounded, as $n \in L$ increases. Here, it is essential that by (7)

$$|y - \eta_{k^*}|, |y - \xi_{k^*+1}| \geqslant \frac{D_4}{2} d_n(k^*), \qquad \text{for all sufficiently large} \quad n \in L. \tag{11}$$

Thus, since $y_{k^*} = \xi_{k^*}$, we have

$$\begin{aligned} |p_n^{k^*}(y)| &\geqslant D\, |y - \xi_{k^*}| \prod_{\substack{\nu \in I_{k^*} \\ \nu \neq k^*}} |\xi_{k^*} - y_\nu| \sum_{l < j < k^*} \frac{1}{\prod_{\nu \in I_{k^*},\, \nu \neq j} |y_j - y_\nu|} \frac{1}{|y_j - y|} \\ &= D \sum_{l < j < k^*} \frac{\prod_{\nu \in I_{k^*},\, \nu \neq k^*} |\xi_{k^*} - y_\nu|}{\prod_{\nu \in I_{k^*},\, \nu \neq j} |\xi_j - y_\nu|} \frac{|y - \xi_{k^*}|}{|y_j - y|}. \end{aligned}$$

2. Next, we show that for some $D > 0$

$$\frac{\prod_{\nu \in I_{k^*},\, \nu \neq k^*} |\xi_{k^*} - y_\nu|}{\prod_{\nu \in I_{k^*},\, \nu \neq j} |\xi_j - y_\nu|} \geqslant D \frac{d_n(j)}{d_n(k^*)}, \qquad \text{for all} \quad l(n) < j < k^*(n),$$

and all sufficiently large $n \in L$.

By the definition of y_j we have

$$\frac{\prod_{v\in I_{k^*},\, v\neq k^*}|\xi_{k^*}-y_v|}{\prod_{v\in I_{k^*},\, v\neq j}|\xi_j-y_v|}$$

$$=\frac{\prod_{v\in I_{k^*},\, v\neq k^*}|\xi_{k^*}-\xi_v|}{\prod_{v\in I_{k^*},\, v\neq j}|\xi_j-\xi_v|}\left(\prod_{\substack{v\in I_{k^*}\\ v<l}}\frac{|\xi_{k^*}-y_v|\,|\xi_j-\xi_v|}{|\xi_{k^*}-\xi_v|\,|\xi_j-y_v|}\right)$$

$$\times\left\{\prod_{\substack{v\in I_{k^*}\\ v>k^*}}\frac{|\xi_{k^*}-y_v|\,|\xi_j-\xi_v|}{|\xi_{k^*}-\xi_v|\,|\xi_j-y_v|}\right\}.$$

It is easy to check that each factor in $(\cdots)$ and $\{\cdots\}$ is greater than or equal to 1, and we obtain

$$\frac{\prod_{v\in I_{k^*},\, v\neq k^*}|\xi_{k^*}-y_v|}{\prod_{v\in I_{k^*},\, v\neq j}|\xi_j-y_v|}\geqslant\frac{\prod_{v\in I_{k^*},\, v\neq k^*}|\xi_{k^*}-\xi_v|}{\prod_{v\in I_{k^*},\, v\neq j}|\xi_j-\xi_v|}.$$

Since

$$\frac{|\xi_j-\xi_{j_{k^*}}|}{|\xi_{k^*}-\xi_{j_{k^*}}|}\geqslant\frac{1}{2},\qquad \text{for all}\quad l(n)<j<k^*(n),$$

and by the definition of k^*, we get that for all $n\in L$

$$\frac{\prod_{v\in I_{k^*},\, v\neq k^*}|\xi_{k^*}-y_v|}{\prod_{v\in I_{k^*},\, v\neq j}|\xi_j-y_v|}$$

$$\geqslant\frac{\prod_{v\in I_{k^*},\, v\neq k^*}|\xi_{k^*}-\xi_v|}{\prod_{v\in I_{k^*},\, v\neq j}|\xi_j-\xi_v|}$$

$$=\frac{|(\xi_{k^*}-\xi_{k^*-1})(\xi_{k^*}-\xi_{k^*+1})|}{|(\xi_j-\xi_{j-1})(\xi_j-\xi_{j+1})|}\,\frac{|\xi_j-\xi_{j_{k^*}}|}{|\xi_{k^*}-\xi_{j_{k^*}}|}$$

$$\times\left(\prod_{\substack{v=1\\ |v-k^*|\geqslant 2}}^{n+2}|\xi_{k^*}-\xi_v|\right)\left(\prod_{\substack{v=1\\ |v-j|\geqslant 2}}^{n+2}|\xi_j-\xi_v|\right)^{-1}$$

$$\geqslant\frac{|(\xi_{k^*}-\xi_{k^*-1})(\xi_{k^*}-\xi_{k^*+1})|}{|(\xi_j-\xi_{j-1})(\xi_j-\xi_{j+1})|}\,\frac{1}{2}\,\frac{d_n(j)^3}{d_n(k^*)^3},\qquad \text{for all}\quad l(n)<j<k^*(n).$$

Lemma 1 and (7) yield that for some $D>0$

$$\geqslant \frac{D_4^2\, d_n(k^*)^2}{C_1^2\, d_n(j)^2}\,\frac{1}{2}\,\frac{d_n(j)^3}{d_n(k^*)^3}$$

$$\geqslant D\,\frac{d_n(j)}{d_n(k^*)}, \qquad \text{for all} \quad l(n)<j<k^*(n),$$

and all sufficiently large $n\in L$.

3. We now have that for some $D>0$

$$|p_n^{k^*}(y)| \geqslant D \sum_{l<j<k^*} \frac{d_n(j)}{d_n(k^*)}\,\frac{|y-\xi_{k^*}|}{|y_j-y|},$$

for all sufficiently large $n\in L$. By Lemma 1 and (11) we obtain

$$\geqslant D \sum_{l<j<k^*} \frac{d_n(j)}{d_n(k^*)}\,\frac{D_4\, d_n(k^*)}{2C_1\, d_n(j,k^*)}$$

$$= D\,\frac{D_4}{2C_1} \sum_{l<j<k^*} \frac{d_n(j)}{d_n(j,k^*)}, \qquad \text{for all sufficiently large} \quad n\in L,$$

and Lemma 2(e) yields

$$\geqslant D\,\frac{D_4}{2C_1}\,(D_2 \log\{k^*(n)-l(n)\} - C_2), \qquad \text{for all sufficiently large} \quad n\in L.$$

Since $\lim_{n\in L} k^*(n)-l(n)=\infty$, it follows that

$$\lim_{n\in L} |p_n^{k^*}(y)| = \infty,$$

which contradicts our principal assumption (3). Hence, Theorem 1 is proven.

3. REMARK

For any subsequence L, where the number of sign components $E_n^1, ..., E_n^{m(n)}$ is exactly $m(n)=n+2$, $n\in L$, we have proved that there exist polynomials $p_n\in P_n$, $n\in L$, satisfying the assumptions of Theorem 2, and such that $\lim_{n\in L}\|p_n\|=\infty$. In this part of the proof only the alternation property of the sign functions σ_n on $E_n^1, ..., E_n^{n+2}$ was used and no further reference was made to the fact that σ_n and $E_n^1, ..., E_n^{n+2}$ are induced by the error functions $f(x)-q_n^*(f,x)$.

Thus, we obtain the following technical Lemma that may be of interest on its own.

LEMMA. *For each* $n \in \mathbb{N}_0$ *let* $E_n^1 < \cdots < E_n^{n+2}$ *be arbitrary compact subsets of* $[-1, 1]$. *Further, for each* $n \in \mathbb{N}_0$, *let* σ_n *be a sign function on* $\bigcup_{j=1}^{n+2} E_n^j$ *such that*

$$\sigma_n|_{E_n^j} = \delta_n(-1)^j, \qquad \text{for all} \quad 1 \leqslant j \leqslant n+2,$$

with some $\delta_n \in \{-1, 1\}$, $n \in \mathbb{N}_0$.

Then there exist polynomials $p_n \in P_n$, $n \in \mathbb{N}_0$, *satisfying*

$$-1 \leqslant \max_{E_n^j} \sigma_n(x)\, p_n(x) \leqslant 1, \qquad \text{for all} \quad 1 \leqslant j \leqslant n+2,$$

such that $\lim_{n \to \infty} \|p_n\| = \infty$.

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